Learning Seminar
Trace formula

Introduction Trace formulas are important tool in number theory, specially in the theory of automorphic forms. These types of formulas can be regarded as non-commutative generalization of the Poissons Summation Formulae.

(Uses)

- Isolated
  - Dimension of automorphic forms
  - Weyl's law
- Comparison
  - Langlands Functoriality.
  - Shimura Varieties

§ Trace formula: Finite groups.

Let G be a finite group and Γ a subgroup of G.

(Induced Representations) Let (σ, W) be a Representation of Γ (i.e. group homomorphism $\sigma : \Gamma \to GL(W)$, where W is a $\mathbb{C}$-vector space).

Consider

$$\mathrm{Ind}_\Gamma^G(\sigma) := \{\varphi : G \to W : \quad \varphi(\gamma \cdot x) = \sigma(\gamma)\varphi(x),\ \gamma \in \Gamma, x \in G\}.$$

Given $g \in G$ and $\varphi \in \mathrm{Ind}_\Gamma^G(\sigma)$, we define

$$R(g)\varphi(x) = \varphi(x \cdot g) \quad , x \in G$$

We obtain a represention of G, called induced rep. from σ to G.

This is the main space that the trace formula studies. For simplicity, we consider the case σ is trivial:

$$\mathrm{Ind}_\Gamma^G(\sigma) = \mathrm{Fct}(\Gamma \backslash G, \mathbb{C}).$$

(Distributions and Decomposition)

Let (π, V) be a representation of G. For $f: G \to \mathbb{C}$, let

$$\pi(f)v := \sum_{g \in G} f(g)\pi(g)\cdot v \quad , \quad v \in V.$$

Rmk · $\pi: \mathrm{Fct}(G, \mathbb{C}) \to \mathrm{End}(V)$ is a distribution (or a generalized function ...)

· Let dg be the counting measure on G, then $\pi(f)v = \int_G f(g)\pi(g)\cdot v \, dg$.

The main object of study of the trace formula is the operator

$$\begin{aligned} R(f) : \mathrm{Ind}_\Gamma^G(\sigma) &\longrightarrow \mathrm{Ind}_\Gamma^G(\sigma) \qquad \underbrace{[\pi(g).\varphi](x)} \\ \varphi &\longmapsto \left(x \mapsto \sum_{g \in G} f(g)\varphi(x \cdot g)\right) \end{aligned}$$

(Spectral decomposition) All finite group representations can be decomposed in irreducible representation. There are finite many non-isomorphic irred rep. of a fixed group G.