(Distributions and Decomposition)

Let (π, V) be a representation of G. For $f: G \to \mathbb{C}$, let

$$\pi(f)v := \sum_{g \in G} f(g)\pi(g)\cdot v \quad , \quad v \in V.$$

Rmk · $\pi : \mathrm{Fct}(G, \mathbb{C}) \to \mathrm{End}(V)$ is a distribution (or a generalized function ...)

· Let dg be the counting measure on G, then $\pi(f)v = \int_G f(g)\pi(g)\cdot v \, dg$.

The main object of study of the trace formula is the operator

$$R(f) : \mathrm{Ind}_\Gamma^G(\sigma) \longrightarrow \mathrm{Ind}_\Gamma^G(\sigma)$$

$$\varphi \longmapsto \Big(x \mapsto \sum_{g \in G} f(g)\varphi(x g)\Big)$$

$[\pi(g).\varphi](x)$

(Spectral decomposition) All finite group representations can be decomposed in irreducible representation. There are finite many non-isomorphic irred rep. of a fixed group G.

- **Irreducible**: V is irreducible if all stable by G-subspace ($\neq \{0\}$) are either $\{0\}$ or V

<u>e.g.</u> $\mathbb{Z}/n\mathbb{Z} \hookrightarrow GL_1(\mathbb{C})$: $t \bmod n \longmapsto (z \longmapsto e^{\frac{2\pi i t}{n}} \cdot z)$

- **Decompose**: For every representation (π, V) ($\neq \{0\}$), there exists irreducible V_i's such that $V \underset{T}{\cong} \bigoplus_i V_i$ (G-equivariant: $T(g \cdot v) = g \cdot T(v)$).

<u>e.g.</u> $\pi : \mathbb{Z}/2\mathbb{Z} \longrightarrow GL_2(\mathbb{C})$. Then $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$

$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad \mathbb{Z}/2\mathbb{Z} : \pi \quad$ Trivial $\quad (z \mapsto -z)$

(Kernel and Traces: Unfolding)

Let $x \in \Gamma \backslash G$, $f : G \longrightarrow \mathbb{C}$ and $\varphi \in \mathrm{Ind}_\Gamma^G(\sigma)$. Let us look at:

$$\sum_{g \in G} f(g) \varphi(x \cdot g) = \sum_{g' \in G} f(x^{-1} g') \varphi(g') \qquad (g' = x \cdot g)$$

$$G = \bigcup_{y \in \Gamma \backslash G} \Gamma \cdot y \qquad = \sum_{y \in \Gamma \backslash G} \sum_{\gamma \in \Gamma} f(x^{-1} \gamma \cdot y) \varphi(\gamma \cdot y) \qquad \text{(unfolding)}.$$

$$= \sum_{y \in \Gamma \backslash G} \sum_{\gamma \in \Gamma} f(x^{-1}\gamma \cdot y)\, \sigma(\gamma) \cdot \varphi(y) \qquad (\text{definition})$$

$$= \sum_{y \in \Gamma \backslash G} \Big(\sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)\, \sigma(\gamma) \Big) \cdot \varphi(y).$$

$$= \sum_{y \in \Gamma \backslash G} \Big(\sum_{\gamma \in \Gamma} f(x^{-1}\gamma y) \Big) \varphi(y) \qquad \lrcorner \qquad (\sigma = 1)$$

Rmk. • $G \times G \longrightarrow \mathbb{C}$

$(x, y) \longmapsto \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)$ is Γ-invariant (left) on both arguments.

We denote the induced function by $K_f : \Gamma \backslash G \times \Gamma \backslash G \longrightarrow \mathbb{C}$ and is called the Kernel of f.

• $\big(K_f(x, y)\big)_{x, y \in (\Gamma \backslash G)^2}$ is the matrix of $R(f)$ in the basis $\{ \delta_z : z \in \Gamma \backslash G \}$

where $\delta_z(y) = \begin{cases} 1 & \text{if } y = z \\ 0 & \text{if } y \neq z \end{cases}$:

Indeed,

$$R(f)\delta_z(x) = \sum_{y\in\Gamma\backslash G} K(x,y)\,\delta_z(y) = K(x,z)$$

$$\therefore\ R(f)\delta_z = \sum_{y\in\Gamma\backslash G} K(y,z)\,\delta_y$$

From the remark above, we see that

$$\operatorname{Tr} R(f) = \sum_{x\in\Gamma\backslash G} K(x,x).$$

e.g. $f=\delta_g \Rightarrow \operatorname{Tr}\pi(g) = \{x\in\Gamma\backslash G : xgx^{-1}\in\Gamma\} = \operatorname{Fix}(g;\Gamma\backslash G)$.

(Trace formula) From the spectral decomposition, we have, After identification, $R(f)V_\pi\subset V_\pi$ and $\operatorname{Tr} R(f) = \sum_{\substack{\pi\\ \text{irred.}}} \operatorname{Tr}\pi(f)$ for all $f: G\to\mathbb{C}$.

We thus have:

$$\sum_{\pi} m(\pi)\operatorname{Tr}\pi(f) = \operatorname{Tr} R(f) = \sum_{x\in\Gamma\backslash G} K(x,x)$$

$$= \sum_{x\in\Gamma\backslash G}\ \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x).$$