Indeed,

$$R(f)\delta_z(x) = \sum_{y\in\Gamma\backslash G} K(x,y)\,\delta_z(y) = K(x,z)$$

$$\therefore\ R(f)\delta_z = \sum_{y\in\Gamma\backslash G} K(y,z)\,\delta_y$$

From the remark above, we see that

$$\operatorname{Tr} R(f) = \sum_{x\in\Gamma\backslash G} K(x,x).$$

e.g. $f=\delta_g \Rightarrow \operatorname{Tr}\pi(g) = \{x\in\Gamma\backslash G : xgx^{-1}\in\Gamma\} = \operatorname{Fix}(g;\Gamma\backslash G)$.

Week 2

(Origami) From the spectral decomposition, we have, after identification, $R(f)V_\pi \subset V_\pi$ and $\operatorname{Tr} R(f) = \sum_{\pi \text{ irred.}} \operatorname{Tr}\pi(f)$ for all $f: G\to\mathbb{C}$.

We thus have:

$$\sum_{\pi} m(\pi)\operatorname{Tr}\pi(f) = \operatorname{Tr} R(f) = \sum_{x\in\Gamma\backslash G} K(x,x)$$

$$= \sum_{x\in\Gamma\backslash G}\ \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x).$$

where $m(\pi)$ is the multiplicity of π in $\operatorname{Ind}_\Gamma^G(\sigma)$.

$\Gamma = \bigcup_{\gamma \in \{\Gamma\}} \{\text{conjugacy class of } \gamma \text{ in } \Gamma.\}$

$\Gamma_\gamma = \{\sigma \in \Gamma;\ \sigma\gamma\sigma^{-1} = \gamma\}.$

$$= \sum_{x \in \Gamma\backslash G} \sum_{\gamma \in \{\Gamma\}} \sum_{\delta \in \Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x) \quad \text{(unfolding)}$$

$$= \sum_{\gamma \in \{\Gamma\}} \sum_{x \in \Gamma\backslash G} \sum_{\delta \in \Gamma_\gamma\backslash\Gamma} f(x\delta^{-1}\gamma\delta x)$$

$$\Gamma_\gamma\backslash G = \bigcup_{x \in \Gamma\backslash G} \Gamma_\gamma\backslash\Gamma \cdot x$$

$$= \sum_{\gamma \in \{\Gamma\}} \sum_{x \in \Gamma_\gamma\backslash G} f(x^{-1}\gamma x) \quad \text{(folding)}$$

$$\Gamma_\gamma\backslash G = \Gamma_\gamma\backslash G_\gamma \Big\backslash G_\gamma\backslash G$$

$$= \sum_{\gamma \in \{\Gamma\}} \frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{x \in G_\gamma\backslash G} f(x^{-1}\gamma x) \quad \text{(unfolding)}$$

$G_\gamma = \{g \in G;\ g\gamma g^{-1} = \gamma\}.$

(Trace formula)

$$\sum_\pi m_\pi \operatorname{Tr}(\pi(f)) = \sum_{\gamma \in \{\Gamma\}} \frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{x \in G_\gamma\backslash G} f(x^{-1}\gamma x)$$

(spectral) (geometric)

("Application": Frobenius Reciprocity) We want to extract information from the trace formula.

Week 3

Rmk Left hand side looks like Fourier Transform!!