Fourier Transform :

$$\mathrm{Fct}(G,\mathbb{C}) \xrightarrow{F} \prod_{\substack{\pi \\ \text{irred.}}} \mathrm{End}(V_\pi) \; ; \; f \longmapsto \left(\frac{1}{|G|}\pi(f)\right)_\pi .$$

e.g. $G = \mathbb{Z}/N\mathbb{Z}$; irred $\Rightarrow$ dimension one $\Rightarrow z \mapsto e^{\frac{2\pi i t}{N} z}$; $t \bmod N$.

- $\prod_\pi \mathrm{End}(V_\pi) = \prod_\pi \mathbb{C} \simeq \mathrm{Fct}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C})$
- $f \longmapsto \mathcal{F}f(n) = \frac{1}{N}\sum_t e^{\frac{2\pi i t n}{N}} f(t)$ one usually put this sign.

Rmk • The spaces $\mathrm{Fct}(G,\mathbb{C})$ and $\prod_{\pi \text{ irred}} \mathrm{End}(V_\pi)$ are Hermitian.

These are, respectively,

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum f_1(g)\overline{f_2(g)} \; ; \; \langle X_1, X_2\rangle = \sum_{\substack{\pi \\ \text{irred}}} \dim V_\pi \cdot \mathrm{Tr}({}^t\overline{X}_\pi, Y_\pi).$$

Thm • F is an isometry.

- If π_0 is an irreducible, then

$$\left(\mathrm{Tr}(\pi_0) : G \to \mathbb{C}\right) \longmapsto \left(\frac{1}{\dim \pi}\,\delta_{\pi_0^*}\right)_\pi \; ; \; \delta_{\pi_0,\pi} = \begin{cases} \mathrm{Id}_{\pi_0} & \pi \simeq \pi_0 \\ 0 & \pi \not\simeq \pi_0 \end{cases},$$

Rmk Let π_1, π_2 be irreducible representations. Then

- $\langle \mathrm{Tr}(\pi_0), \mathrm{Tr}(\pi_1) \rangle = \begin{cases} 1 & \text{if } \pi_0 \simeq \pi_1 \\ 0 & \text{if } \pi_0 \not\simeq \pi_1 \end{cases}$

- If (V, π) is any (finite dimensional) representation, then $\frac{1}{\# G} \sum_{g \in G} \mathrm{Tr}\, \pi(g) = \dim V^G$

  $= \langle \mathrm{Tr}(\mathrm{triv}), \mathrm{Tr}\, \pi \rangle$

Indeed, • $\dim V^G = \#$ of trivial Rep in the decomposition of V.

• $\mathrm{Tr}\, \pi = \sum_i \mathrm{Tr}\, \pi_i$

Apply $f = \overline{\mathrm{Tr}\, \pi_0}$

$$m_{\pi_0} |G| = \sum_{\gamma \in \{\Gamma\}} \frac{|\cancel{G_\gamma}|}{|\Gamma_\gamma|} \frac{|G|}{|\cancel{G_\gamma}|} \overline{\mathrm{Tr}\, \pi_0(\gamma)}$$

$$= |G| \sum_{\gamma \in \{\Gamma\}} \frac{1}{|\Gamma_\gamma|} \overline{\mathrm{Tr}\, \pi_0(\gamma)} = |G| \overline{\sum_{\gamma \in \Gamma} \mathrm{Tr}\, \pi_0(\gamma)}$$

$$= |G| \dim V_0^\Gamma.$$ Remark applied to $\pi_0|_\Gamma$.

Thus, $m_{\pi_0} = \dim V_0^\Gamma$. (Frobenius)

$\boxed{\text{Week 4}}$