Traces & Kernels

We describe the framework to extend from finite to locally compact case.

(Traces).

Let $(H, \langle , \rangle)$ be a separable Hilbert space. For $v \in H$, let $\|v\| = \sqrt{\langle v, v \rangle}$ be the Norm. We want to describe properties of bounded linear maps:

$$T : H \to H. \qquad \text{(continuous)}$$

e.g

- $H = L^2(S^1)$. Let $a \in \mathbb{R}$, $T_a : \begin{matrix} H & \longrightarrow & H \\ \in & & \in \\ \varphi & \mapsto & \varphi(x+a) \end{matrix}$

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(Compact) T is compact if the closure of the image of the unit ball $\to \{v : \|v\| \leq 1\}$ is compact.

Prop T is self-adjoint compact operator, then there is an orthonormal basis for H consisting of eigenvectors for T. The eigenspaces of non-zero eigenvalues are finite dimensional.

e.g. • ~~$H = L^2(\mathbb{R})$, $T_a : H \to H$ is compact~~ False!!

• (non) $H = L^2(\mathbb{R})$, $T_a : H \to H$ is not compact

• $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $e_i \mapsto \lambda_i e_i$, $\lambda_i \to 0$.

• $f \mapsto xf$ is self-adjoint, but doesn't have eigenvectors.

(Polar decomposition) First observe that T^*T is a positive (i.e. $\langle Tv, v\rangle \geq 0, \forall v$) self-joint bounded operator. Using "spectral calculus" we can define:

$$|T| = \sqrt{TT^*}$$

$\|V(x)\| = \|x\|$ at $x \in (\ker V)^\perp$

Moreover, $T = V|T|$ and $|T| = V^*T$. ; where V is a partial isometry

Existence of V:

* $\| |T|(x) \| = \|T(x)\|$
* $\ker |T| = \ker (T^*T) = \ker T$

(Trace class) We say T is trace-class if there exists a orthonormal basis $\{e_i\}$ such that $\sum \langle |T| e_i, e_i \rangle < \infty$.

Rmk Let $\{x_n\}$ be another orthonormal basis of H.

$$\sum_n \langle |T| x_n, x_n \rangle = \sum_n \sum_i \langle \langle x_n, e_i \rangle |T| e_i, x_n \rangle \quad (x_n = \sum \langle x_n, e_i \rangle e_i)$$

$$= \sum_i \sum_n \langle |T| e_i, \langle e_i, x_n \rangle x_n \rangle \quad (\text{Positive sum})$$

$$= \sum_i \langle |T| e_i, e_i \rangle .$$

Prop. Suppose that T is trace class.

i) $\forall S : H \to H$ bounded, ST and TS are trace-class (*-Ideal.)

ii) T is compact

iii) For all orthonormal basis $\{x_n\}$, $\sum \langle T x_n, x_n \rangle$ converges absolutely and it is independent of $\{x_n\}$. We denote this value $\mathrm{Tr}(T)$.

Question: How can we get these type of operator?