(Trace class) We say T is trace-class if there exists a orthonormal basis $\{e_i\}$ such that $\sum \langle |T| e_i, e_i \rangle < \infty$.

Rmk Let $\{x_n\}$ be another orthonormal basis of H.

$$\sum_n \langle |T| x_n, x_n \rangle = \sum_n \sum_i \langle \langle x_n, e_i \rangle |T| e_i, x_n \rangle$$ ($x_n = \sum \langle x_n, e_i \rangle e_i$)

$$= \sum_i \sum_n \langle |T| e_i, \langle e_i, x_n \rangle x_n \rangle$$ (Positive sum)

$$= \sum_i \langle |T| e_i, e_i \rangle .$$

Prop. Suppose that T is trace class.

i) $\forall S : H \to H$ bounded, ST and TS are trace-class (*-Ideal.)

ii) T is compact

iii) For all orthonormal basis $\{x_n\}$, $\sum \langle T x_n, x_n \rangle$ converges absolutely and it is independent of $\{x_n\}$. We denote this value $\mathrm{Tr}(T)$.

Question: How can we get these type of operator?

Extra (Key: Cauchy-Schwarz inequality)

Proof (Idea) i) Reduction ⇝ self-adjoint ⇝ unitary $\Rightarrow |S| = 1$

$\Rightarrow |ST| = |T|.$ ✓

ii) $|T|^2$ is trace class, then

$$\sum \langle |T|^2 e_i, e_i \rangle = \sum \langle T^* T e_i, e_i \rangle$$
$$= \sum_i \| T e_i \|^2 < \infty.$$

For $N \in \mathbb{N}$, Consider

$$T_N = \sum_{i=1}^{N} \langle -, e_i \rangle T(e_i).$$

The T_N's are compact. On the other hand, we note

$$| T(v) - T_N(v) | \leq \sum_{i=N+1}^{\infty} |\langle v, e_i \rangle \cdot T(e_i)|$$
$$\leq \left(\sum_{i=N+1}^{\infty} |\langle v, e_i \rangle|^2 \right)^{1/2} \cdot \left(\sum \| T(e_i) \|^2 \right)^{1/2}$$
$$\leq \|v\| \cdot \left(\sum_{i\, N+1}^{\infty} \| T(e_i) \|^2 \right)^{1/2} \xrightarrow{N \to \infty} 0$$

iii) $T = V\,|T|^{1/2}\,|T|^{1/2}$

$$|\langle T e_i, e_i\rangle| = |\langle |T|^{1/2} e_i, |T|^{1/2} V^* e_i\rangle|$$

$$\leq \underbrace{\| |T|^{1/2} U^* e_i \|}_{= \sqrt{\langle |T| U U^* e_i, e_i\rangle}} \; \underbrace{\| |T|^{1/2} e_i \|}_{= \sqrt{\langle |T| e_i, e_i\rangle}} \qquad (|T|^{1/2})^* = |T|^{1/2}$$

$$\Rightarrow \sum_i |\langle T e_i, e_i\rangle| \leq \Big(\sum \langle |T| U U^* e_i, e_i\rangle\Big)^{1/2} \Big(\sum \langle |T| e_i, e_i\rangle^2\Big)^{1/2}$$

$|T| U U^*$ & $|T|$ are trace class.

Independence: as in remark.

• (Hilbert-Schimdt) T is Hilbert-Schimdt (H-S) if there is an orthonormal basis $\{e_i\}$ such that

$$\sum_i \|Te_i\|^2 < \infty$$

Prop i) H-S $\Rightarrow$ compact • similar, As we did before (Cauchy-Schwarz)

ii) Trace-class $\Rightarrow$ H-S • $\|Te_i\|^2 = \langle Te_i, Te_i \rangle = \langle T T^* e_i, e_i \rangle = \langle |T|^2 e_i, e_i \rangle.$

iii) H-S $\circ$ H-S $\Rightarrow$ Trace-class

Proof iii) write $ST = |ST| V$, V is a partial isometry. So, $|ST| = V^* ST$. Then $\langle |ST| e_i, e_i \rangle = \langle V^* ST e_i, e_i \rangle = \langle Te_i, S^* V e_i \rangle$. Observe that $\{Ve_i : Ve_i \neq 0\}$ is orthonormal, since V is a partial isometry. Using this, we have

$$\underbrace{\sum \langle |ST| e_i, e_i \rangle}_{\geq 0} = \sum |\langle Te_i, S^* V e_i \rangle| \leq \Big(\underbrace{\sum_i \|Te_i\|^2}_{<\infty}\Big)^{1/2} \cdot \Big(\underbrace{\sum_i \|S^* V e_i\|^2}_{<\infty}\Big)^{1/2}$$

(Kernels) We study closely the case $H = L^2(X,\mu)$, where (X,μ) is a measured space. Let $K \in L^2(X\times X, \mu\times\mu)$, then

$$\begin{aligned} T_K : L^2(X,\mu) &\longrightarrow L^2(X,\mu) \\ \varphi &\longmapsto \left(x \longmapsto \int K(x,y)\varphi(y)\,d\mu(y)\right) \end{aligned}$$

is Hilbert-Schmidt. Indeed,

$$\sum_i \|A_K e_i\|^2 \underset{\text{Parseval}}{=} \sum_i \sum_j |\langle A_K e_i, e_j\rangle|^2 = \sum_{i,j} |\langle K(x,y), e_i(y)\overline{e_j(x)}\rangle_{L^2(X\times X)}|^2$$

(Pythagoras ?!)

$$\leq \|K(x,y)\|^2_{L^2(X\times X)}.$$

$\{(x,y) \mapsto e_i(y)\overline{e_j(x)}\}$ is orthonormal in $L^2(X\times X)$

Moreover, $HS(X,\mu) = \{$Hilbert-Smith operator of $L^2(X,\mu)\}$, then

- $HS(X,\mu)$ is a Hilbert space: $\langle T, S\rangle = \mathrm{Tr}(TS^*)$.

- $$\begin{aligned} HS(X,\mu) &\longrightarrow L^2(X\times X, \mu\times\mu) \\ A_K &\longleftarrow K \end{aligned}$$ is an isometry.