Automorphic form on GL_2

Let $\Gamma = SL_2(\mathbb{Z})$. Let $M_m(\Gamma)$ = modular form of wt. m

$= f: \mathcal{H} \to \mathbb{C}$, holomorphic such that,

$$f(\gamma z) = J(\gamma; z)^m f(z), \quad \gamma \in \Gamma$$

where $J(g; z) = \det(g)^{-1/2}(cz+d)$,

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$.

$\mathcal{H} = GL(2,\mathbb{R})^+ \cdot i$ through mobius transformation.

the $\mathrm{Stab}(i)$

$= Z \cdot K_\infty^+$.

where $Z = \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} : z > 0 \right\}$, $K_\infty^+ = SO(2)$

$$\begin{array}{ccc} GL(2,\mathbb{R})^+ & \xrightarrow{F} & \mathbb{C} \\ \downarrow & \nearrow_{f} & \\ \mathcal{H} & & \end{array}$$

If we define $F(g) = f(g \cdot i)$. then it satisfies, $F(zgk) = F(g)$.

Moreover we set $\varphi(g) = J(g; i)^{-m} F(g) = J(g; i)^{-m} f(g \cdot i)$

Then $\varphi(g)$ satisfies.

i) $\varphi(\gamma g) = \varphi(g)$, $\forall \gamma \in \Gamma$ and $g \in GL_2^+(\mathbb{R})$

ii) $\varphi(zg) = \varphi(g)$ for all $z \in Z$.

iii) $\varphi(g k_\theta) = e^{\pi i m \theta} \varphi(g)$,

$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_\infty^+$.

iv) $\varphi(g)$ is an eigenfunction for $Z(\mathfrak{g}_\mathbb{C}) \to$ center of the universal enveloping algebra.

[relates to holomorphicity of f).

v) $\varphi(g)$ is of moderate growth.

$\exists\, n \in \mathbb{Z}_+$ such that $|\varphi(g)| \le C \|g\|^n$.

$\varphi(g)$ is an automorphic form on $GL(2,\mathbb{R})^+$

We now define Automorphic form on the Adelic version $GL(2,\mathbb{A}_{\mathbb{Q}})$.

$$\begin{tikzcd} G(\mathbb{Q})\backslash G(\mathbb{A}) \arrow[d] \arrow[ddr, "\varphi"] & \\ Z(\mathbb{A}_{\mathbb{Q}})G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/K_0(N) & \\ Z(\mathbb{R})\Gamma_0(N)\backslash GL(2,\mathbb{R})^{+} \arrow[u, equal] \arrow[r, "\varphi"] & \mathbb{C} \end{tikzcd}$$

We define any smooth function φ on $GL(2,\mathbb{A}_{\mathbb{Q}})$ as the following: $\varphi(g) := \varphi(g_\infty, g_f)$, where $g_\infty \in \mathbb{Q}_{\infty}$, $g_f \in \mathbb{A}_f$.

and it is C^∞ in the variable g_∞ (Arch.) and locally constant in the variable g_f (non-Arch.).

Moreover it will satisfy:

i) $\varphi(\gamma g) = \varphi(g)$ $\forall \gamma \in G(\mathbb{Q})$

ii) $\varphi(g k_0 k_f) = e^{im\theta}\varphi(g)$, (more generally $\langle \varphi(gk) \mid k \in K\rangle$ is f.d.)

iii) $\langle X\varphi(g) \mid X \in Z(\mathfrak{gl}_{\mathbb{C}})\rangle$ is finite dimensional.

iv) there exists $c, r > 0$ such that

moderate growth

$|\varphi(g)| \leq c\,\|g\|^{r}$, where $\|g\| = \prod_{v} \left(\max_{i,j}\{|g_{ij}|_v, |(g^{-1})_{ij}|_v\}\right)$

iii can be equivalently described as: there exists an ideal $J \subset Z(\mathfrak{g})$ of finite co-dimension such that $J\cdot\varphi = 0$.

Let G be a reductive group / global field F. Then we have the following theorem of Harish-chandra

$\underline{\text{Th}^m}$:→ If we fix δ be a f.d. rep. of K_∞, $L \subset K_f$, an open compact subgroup, $J \subset Z(\mathfrak{g})$ an ideal of finite co-dimension, and $\omega : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ a central character. Let $A(\delta, L, J, \omega)$ denote the set of automorphic forms such that.

i) φ transforms by δ under K_∞.

ii) $\varphi(gl) = \varphi(g)$ for all $l \in L$.

iii) $J.\varphi = 0$

iv) $\varphi(zg) = \omega(z)\varphi(g)$, $z \in Z(\mathbb{A}_F)$.

Then $\dim_{\mathbb{C}} A(\delta, L, J, \omega) < \infty$.

$\boxed{\text{Smooth Automorphic Forms}}$:→

Now Automorphic forms defined above <u>do not</u> get preserved under the right translation action of $G(\mathbb{A}_F)$. Let φ is an automorphic form. Let $\varphi'(g) = R(g')\varphi(g) = \varphi(gg')$.

Then φ is Right K-finite $\Rightarrow$ φ' is $\underline{g' K (g')^{-1}}$ - finite.

Now we know $K = K_\infty K_f$, let $K' = g' K (g')^{-1}$. At finite places, K_f and K_f' are both open and compact and $K_f \cap K_f'$ has finite index in both. Therefore there are no differences between K_f-finiteness and K_f'-finiteness.

But φ' would be $g_\infty K_\infty g_\infty^{-1}$ - finite.

Hence it will depend on the choice of K_∞.

Therefore $\mathcal{A}$ is $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_F^\infty)$-module, rather than $G(\mathbb{A}_F)$-module.

To make it $G(\mathbb{A}_F)$-module we enlarge the space $\mathcal{A}$ to $\mathcal{A}^\infty$ by weakening K_∞-finite condition to uniform moderate growth which states:

If φ is smooth and there exists $r \in \mathbb{R}$, such that for all $X \in \mathcal{U}(\mathfrak{g}_\mathbb{C}) \Rightarrow |X\varphi(g)| \leq C_X \|g\|^r$ for some $C_X \in \mathbb{R}_{>0}$.

Now through the above definition $\mathcal{A} \subset \mathcal{A}^\infty$, where $\mathcal{A}$ is the precise space of $\varphi \in \mathcal{A}^\infty$ that are K-finite, and in fact $\mathcal{A}$ is dense in $\mathcal{A}^\infty$.

L^2-Automorphic forms

We must fix an unitary central character $\omega: F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^!$

Then
$$L^2(\omega) = L^2\left(G(F)\backslash G(\mathbb{A}_F), \omega\right)$$
$$:= \left\{ \varphi : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C},\ \varphi(zg) = \omega(z)\varphi(g) \right\}$$

such that
$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A}_F)} |\varphi(g)|^2 \, dg < \infty.$$

$G(\mathbb{A}_F)$ acts on $L^2(\omega)$ by right translation which preserves the norm, hence $L^2(\omega)$ affords a unitary rep. of $G(\mathbb{A}_F)$.

A classical modular form $f(z)$ for $\Gamma = SL_2(\mathbb{Z})$ is called a cusp form if

$$0 = a_0 = \int_0^1 f(x+iy)\,dx = \int_0^1 f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} iy\right)dx$$

If $\varphi(g)$ is an automorphic form on GL_2, then the group of translations is

$$N = \left\{ n = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \right\} \subset GL_2.$$

So the analogous integral would be over $\dfrac{N_2(\mathbb{A})}{N(k)} \cong \dfrac{\mathbb{A}}{k}$, hence compact.

Def^n $\to$ An automorphic form $\varphi(g)$ on $GL_2(\mathbb{A})$ is a cusp form iff

$$\int_{N_2(\mathbb{A})/N_2(k)} \varphi(ng)\,dg = \int_{\mathbb{A}/k} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)dx = 0.$$

Now for $\underline{GL_n(\mathbb{A})}$, the above condition is replaced by a family of integrals which are given by the unipotent radicals of rational parabolic subgroups. The parabolic subgroups are parametrized upto conjugation by partitions $n = n_1 + n_2 + \cdots + n_r$. The associated parabolic is

$$P = \left\{ \begin{pmatrix} g_1 & * & \cdots & * \\ & g_2 & & \vdots \\ & & \ddots & * \\ & & & g_r \end{pmatrix} \;\middle|\; g_i \in GL_{n_i} \right\} = M \cdot U,$$

where $M \cong GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r}$, $U \cong \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ & I_{n_2} & & \vdots \\ & & \ddots & * \\ & & & I_{n_r} \end{pmatrix} \right\}$

An Automorphic form $\varphi(g)$ on $GL_n(\mathbb{A})$ is a cusp form iff $\displaystyle\int_{\frac{U(\mathbb{A})}{U(k)}} \varphi(ug)\,du = 0$, for all unipotent radicals of all proper parabolic subgroups of GL_n.

All parabolic subgroups are conjugate to a standard parabolic, hence it is enough to consider integrals over the standard unipotent radicals. In fact it suffices to consider only unipotent radicals of maximal parabolic subgroups, i.e. $n = n_1 + n_2$, so

$$P = \left\{ \begin{pmatrix} g_1 & x \\ & g_2 \end{pmatrix} \right\} \supset U = \left\{ \begin{pmatrix} I_{n_1} & x \\ & I_{n_2} \end{pmatrix} \right\} \cong M_{n_1 \times n_2}.$$

<u>Theorem</u> (Gelfand, Graev-Piatetski Shapiro)

If $\varphi(g)$ is a cusp form, then it is rapidly decreasing for the center of a fundamental domain $\mathcal{F}$ for modulo the center of $\frac{GL_n(\mathbb{A})}{GL_n(k)}$, i.e., for some r,

$$|\varphi(zg)| \le C|z|^{r} \|g\|^{-N}.$$