9

Hence it will depend on the choice of K_∞.

Therefore $\mathcal{A}$ is $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_F^\infty)$-module, rather than $G(\mathbb{A}_F)$-module.

To make it $G(\mathbb{A}_F)$-module we enlarge the space $\mathcal{A}$ to $\mathcal{A}^\infty$ by weakening K_∞-finite condition to uniform moderate growth which states:

If φ is smooth and there exists $r \in \mathbb{R}$, such that for all $X \in \mathcal{U}(\mathfrak{g}_\mathbb{C}) \Rightarrow |X\varphi(g)| \leq C_X \|g\|^r$ for some $C_X \in \mathbb{R}_{>0}$.

Now through the above definition $\mathcal{A} \subset \mathcal{A}^\infty$, where $\mathcal{A}$ is the precise space of $\varphi \in \mathcal{A}^\infty$ that are K-finite, and in fact $\mathcal{A}$ is dense in $\mathcal{A}^\infty$.

L^2- Automorphic forms

We must fix an unitary central character $\omega: F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$!

Then $L^2(\omega) = L^2(G(F)\backslash G(\mathbb{A}_F), \omega)$

$$:= \{\varphi : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C},\ \varphi(zg) = \omega(z)\varphi(g)\}$$

such that $\displaystyle\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A}_F)} |\varphi(g)|^2\, dg < \infty.$

$G(\mathbb{A}_F)$ acts on $L^2(\omega)$ by right translation, which preserves the norm, hence $L^2(\omega)$ affords a unitary rep. of $G(\mathbb{A}_F)$.

A classical modular form $f(z)$ for $\Gamma = SL_2(\mathbb{Z})$ is called a cusp form if
$$0 = a_0 = \int_0^1 f(x+iy)\,dx = \int_0^1 f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot iy\right) dx$$

If $\varphi(g)$ is an automorphic form on GL_2, then the group of translations is
$$N = \left\{ n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\} \subset GL_2.$$

So the analogous integral would be over $\dfrac{N_2(\mathbb{A})}{N(k)} \cong \dfrac{\mathbb{A}}{k}$, hence compact.

Def^n $\rightarrow$ An automorphic form $\varphi(g)$ on $GL_2(\mathbb{A})$ is a cusp form iff
$$\int_{N_2(\mathbb{A})/N_2(k)} \varphi(ng)\,dg = \int_{\mathbb{A}/k} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0.$$

Now for $GL_n(\mathbb{A})$, the above condition is replaced by a family of integrals which are given by the unipotent radicals of rational parabolic subgroups. The parabolic subgroups are parametrized upto conjugation by partitions $n = n_1 + n_2 + \cdots + n_r$. The associated parabolic is
$$P = \left\{ \begin{pmatrix} g_1 & * & \cdots & * \\ & g_2 & & \vdots \\ & & \ddots & * \\ & & & g_r \end{pmatrix} \;\middle|\; g_i \in GL_{n_i} \right\} = M \cdot U,$$
where $M \cong GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r}$, $U \cong \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ & I_{n_2} & & \vdots \\ & & \ddots & * \\ & & & I_{n_r} \end{pmatrix} \right\}$

An Automorphic form $\varphi(g)$ on $GL_n(\mathbb{A})$ is a cusp form iff $\displaystyle\int_{U(\mathbb{A})/U(k)} \varphi(ug)\,du = 0$, for all unipotent radicals of all proper parabolic subgroups of GL_n.

All parabolic subgroups are conjugate to a standard parabolic, hence it is enough to consider integrals over the standard unipotent radicals. In fact it suffices to consider only unipotent radicals of maximal parabolic subgroups, i.e. $n = n_1 + n_2$, so

$$P = \left\{ \begin{pmatrix} g_1 & X \\ & g_2 \end{pmatrix} \right\} \supset U = \left\{ \begin{pmatrix} I_{n_1} & X \\ & I_{n_2} \end{pmatrix} \right\} \cong M_{n_1 \times n_2}.$$

<u>Theorem</u> (Gelfand, Graev-Piatetski Shapiro)

If $\varphi(g)$ is a cusp form, then it is rapidly decreasing modulo the center of a fundamental domain $\mathcal{F}$ for $GL_n(\mathbb{A})/GL_n(k)$, i.e., for some r,

$$|\varphi(zg)| \leq C|z|^r \|g\|^{-N}.$$

Hecke Algebra

12

Global Hecke Algebra = Restricted direct product of local Hecke algebra.

$\mathcal{H} = \bigotimes' \mathcal{H}_v$, $\mathcal{H}$ & $\mathcal{H}_v$ are idempotented algebras under convolution.

Hence there will be a family of ~~on~~ fundamental idempotents $\{\xi_v\}$ such that

$$\mathcal{H} = \bigcup_i \xi_i * \mathcal{H} * \xi_i \quad \text{and} \quad \mathcal{H}_v = \bigcup_i \xi_{i,v} * \mathcal{H} * \xi_{i,v}$$

⊙ If $v \nmid \infty$, non-archimedean place of k (global number field) then $\mathcal{H}_v = C_c^\infty(G(k_v))$ is the algebra of smooth (locally compact) compactly supported functions on $G(k_v)$. For each $L_v \subset k_v$ open compact define the fundamental idempotent

$$\xi_{L_v} = \frac{1}{\mathrm{Vol}(L_v)} \chi_{L_v},$$ where χ_{L_v} is the characteristic function of L_v.

Then $\xi_{L_v} * \mathcal{H}_v * \xi_{L_v} = C_c^\infty(G(k_v)//L_v) \rightarrow$ Bi-L_v invariant function.

ξ_v^0 = fundamental idempotent associated with maximal compact subgroup K_v.

⊙ If $v | \infty$, archimedean place of k, then $\mathcal{H}_v$ is the convolution algebra of bi-K_v-finite distribution on $G(k_v)$ with support in K_v.

Then $\mathcal{H}_v = \mathcal{U}(\mathfrak{g}_v) \otimes \mathcal{U}(k_v)^{A(k_v)} \rightarrow$ finite measures on K_v.

For every finite dimensional rep. of δ_v of k_v, we have a fundamental idempotent

$$\xi_{\delta_v} = \frac{1}{\deg(\delta_v)} \Theta_{\delta_v}\, dk_v, \quad \Theta_{\xi_v} \rightarrow \text{character of } \delta_v.$$

13

(•) $\mathcal{H} = \bigotimes_v' \mathcal{H}_v$

$$= \bigcup_S \left(\bigotimes_{v \in S} \mathcal{H}_v \otimes \left(\bigotimes_{v \notin S} \xi_v^{0} \right) \right)$$

S = finite set of places containing of all Archimedean places

Representation on Automorphic forms (A) :→ The space $\mathcal{A}$ of k-finite automorphic forms is naturally an $\mathcal{H}$-module by right convolution.

For $\xi \in \mathcal{H}$ and $\varphi \in \mathcal{A}$ we define $R(\xi)\varphi := \int_{G(\mathbb{A})} \varphi(gh)\xi(h)\,dh$

Right regular representation

then k-finiteness condition

$\Rightarrow \exists$ an idempotent $\xi = \xi_\infty \otimes \xi_f = \xi_S \otimes \xi_L$ such that $R(\xi)\varphi = \varphi$.

Admissible rep.

A representation (π_v, V_v) of a local Hecke Algebra $\mathcal{H}_v$ is admissible if for every fundamental idempotent ξ_v we have $\dim_{\mathbb{C}} (\pi_v(\xi_v) V_v) < \infty$.

Similarly, a rep. (π, V) of the global Hecke Algebra $\mathcal{H}$ is admissible if for every global fundamental idempotent $\xi \in \mathcal{H}$ the subspace $\pi(\xi)V$ is finite-dimensional.

Definition :→ An automorphic rep. (π, V) of $\mathcal{H}$ is an irreducible (Hence admissible) sub-quotient $\mathcal{A}(G(k)\backslash G(\mathbb{A}))$.

Theorem (Decomposition theorem): If (π, V) is an irreducible admissible rep. of $\mathcal{H}$ then for each place v of k there exists an irreducible admissible representation (π_v, V_v) of $\mathcal{H}_v$, having a K_v-fixed vector for almost all v, such that

$$\pi = \otimes' \pi_v .$$

Corr If (π, V) is an automorphic representation, then π decomposes into a restricted tensor product of local irreducible admissible representations: $\pi = \otimes' \pi_v$

Smooth Automorphic representation $(\mathcal{A}^\infty)$.

A smooth automorphic representation (π, V) of $G(\mathbb{A})$ is a (closed) irreducible subquotient of $\mathcal{A}^\infty(G(k)\backslash G(\mathbb{A}))$.

$$= \varinjlim_{L} \varinjlim_{r} (A_r^\infty)^L$$

$L \subset K_f$ open compact

Smooth $\Rightarrow$ Admissible

(in this case admissible means, for an any (π, V) smooth, its dense subspace of K-finite vectors V_K is admissible as an $\mathcal{H}$-module

15

Theorem (Decomposition theorem in $\mathcal{A}^\infty$)

If (π, V) is a smooth automorphic representation of $G(\mathbb{A})$ then there ~~exists~~ irreducible (adm) smooth rep. (π_v, V_v) of $G(k_v)$, which are smooth Fréchet representations of moderate growth if $v \mid \infty$, such that

$\pi = \pi_\infty \otimes \pi_f$, $\quad \pi_\infty = \widehat{\bigotimes_{v \mid \infty}} \pi_v \to$ topological completion of tensor product of smooth Frechet rep.

$\pi_f = \bigotimes'_{v<\infty} \pi_v \to$ restricted product of smooth rep. of $G(k_v)$.

Moreover, if (π_K, V_K) is the associated $\mathcal{H}$-module of K-finite vectors in V then in the decomposition $\pi_K = \bigotimes' (\pi_K)_v$, we have $\pi_v = (\pi_K)_v$ for $v<\infty$, while for $v \mid \infty$ we have $(\pi_v)_K = (\pi_K)_v$, and $\pi_v = \widehat{(\pi_K)}_v$ is the Casselman-Wallach canonical completion of $\mathcal{H}_v$-module $(\pi_K)_v$.

L^2 Automorphic Representations

fix an unitary central character $\omega : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$.

Definition :→ An L^2-Automorphic representation (π, V) is an irreducible constituents in the L^2-decomposition of some $L^2(\omega)$.

Theorem (Decomposition theorem in L^2) :→ If (π, V) is an irred. unitary L^2 automorphic rep. then there exists rep. (π_v, V_v) of $G(k_v)$ such that $\pi = \widehat{\bigotimes}' \pi_v$ is the restricted Hilbert tensor product.

Cuspidal Representations

Condition : $\displaystyle\int_{\underline{U(\mathbb{A})}\atop U(K)} \varphi(ug)\,du = 0$, closed condition.

The actions of $\mathcal{H}$ / $G(\mathbb{A})$ on the spaces of automorphic forms are by right convolutions / right translations (respectively).

Gelfand - Graev - Piatetski-Shapiro :

The Space $L_0^2(\omega)$ of L^2-cusp forms decomposes into a discrete Hilbert direct sum with finite multiplicities of irreducible unitary sub-representations

$$L_0^2(\omega) = \bigoplus m(\pi) V_\pi \text{ . with } m(\pi) < \infty.$$

Defn :→ The irreducible constituents (π, V_π) of the various $L_0^2(\omega)$ are the L^2-cuspidal representations.