Proposition: Gelfand, Graev, and Piatetski-Shapiro

Let $\varphi \in C_c^\infty(GL(n, \mathbb{A}_F))$

i) There exists a $C > 0$ (depending on φ) such that

$$\sup_{g \in GL(n,\mathbb{A})} |\rho(\varphi) f(g)| \leq C \|f\|_2,$$

$\forall f \in L^2_{cusp}(GL(n,F)\backslash GL(n,\mathbb{A}), \omega)$.

ii) The operator $\rho(\varphi)$ is compact on $L^2_{cusp}(GL(n,F)\backslash GL(n,\mathbb{A}))$.

Pf. $n=2$, $F=\mathbb{Q}$ $\leftarrow$ Proof.

Let $K = \prod_v K_v$, $K_\infty = SO(2)$, $K_v = GL(2, \mathcal{O}_v)$. maximal compact.

$\mathfrak{G}_{c,d} \subset GL(2,\mathbb{A})$ be the set of Adeles, of the form (g_v),

$$g_\infty = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} k_\infty, \quad z \in \mathbb{R}^\times, \ c \leq y, \ 0 \leq x \leq d, \ k_\infty \in K_\infty.$$

$\times$ $g_v \in K_v$ for all non-arch. v.

$\overline{\mathfrak{G}}_{c,d}$ the image of $\mathfrak{G}_{c,d}$ in $Z(\mathbb{A})\backslash GL(2,\mathbb{A})$, if $c \leq \sqrt{3}/2$ and $d \geq 1$, then one can show that,

$$GL(2,\mathbb{A}) = GL(2,\mathbb{Q})\mathfrak{G}_{c,d}.$$

$$(\rho(\varphi)f)(g) = \int_{GL(2,\mathbb{A})} \varphi(h) f(gh)\, dh$$

$$= \int_{Z(\mathbb{A})\backslash GL(2,\mathbb{A})} \underbrace{\int_{\mathbb{A}^\times} \varphi\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} h\right) \omega(z)\, dz}_{\mapsto\ \varphi_\omega(h)} f(gh)\, dh.$$

$$= \int_{Z(\mathbb{A})\backslash GL(2,\mathbb{A})} \varphi_\omega(h) f(gh)\, dh.$$

One can show that $L^2_{cusp}(GL(n,F)\backslash GL(n,A), \omega)$ is closed under the action of $P(\varphi)$.

[Let $V_P = \{ \varphi \in L^2(Z(A)\, GL(n,F)\backslash GL(n,A), \omega) \mid P_{\cancel{x}}(R(g)\varphi) := \int_{Z(A)N_P(F)\backslash Z(A)N_P(A)} R(g)\varphi(n)\, dn = 0 \}$.

$$L^2_{cusp}(GL(n,F)\backslash GL(n,A), \omega) = \bigcap_P V_P$$

$\{$ = conjugacy classes of parabolic subgroup $\}$.

each $f \in C_c^\infty(GL(n,A))$, gives rises to a linear form. V_P's are kernels of this linear form.]

$$\left[Pe_{f,\chi}(g) = \int_{A_G H(F)\backslash A_G H(A_F)} K_f(g,h)\chi^{-1}(h)\, dh \right]$$

Therefore $P(\varphi) f$ is also a cusp form!

~~Therefore~~

$$\left(P(\varphi) f\right)(g) = \int_{N(F)Z(A)\backslash GL(2,A)} K'(g,h)\, f(h)\, dh.$$

where $K'(g,h) = K(g,h) - K_0(g,h)$

$$K(g,h) = \sum_{\gamma \in N(F)} \varphi_\omega(g^{-1}\gamma h), \quad K_0(g,h) = \int_{F\backslash A} \varphi_\omega\left(g^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right) dx.$$

[$F\backslash A$ is compact, integral make sense].

Let $g \in \mathcal{G}_{c,d}$. We write $g = \begin{pmatrix} n & \\ & n \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k_g$.

$x, y \in \mathbb{R}$, $c \leq y$, $0 \leq x \leq d$, $k_g \in K = \prod K_v$

Let $h \in GL(2, A)$, we write $h = \begin{pmatrix} \xi & \\ & \xi \end{pmatrix}\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} k_h$.

Where, $v \in A^\times$, $u \in A$, $k_h \in K$.

By the poisson summation formula,

$$K'(g,h) = \sum_{\alpha \in F^\times} \hat{\Phi}_{g,h}(\alpha), \quad \text{where}$$

$\Phi_{g,h} : A \to \mathbb{C}$ is the compactly supported continuous function

$$\Phi_{g,h}(x) = \varphi_\omega\left(g^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right)$$

When $\alpha = 0$, it is ~~[illegible]~~ same as ~~[illegible]~~ K_0.

$$\hat{\Phi}_{g,h}(\alpha) = \int_A \varphi_\omega\left(g^{-1}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} h\right)\psi(\alpha t)\, dt,$$

Where $\psi : A_F \to \mathbb{C}^\times$ character.

$$= \int_A \varphi_\omega\left(k_g^{-1}\begin{pmatrix} n^{-1}\xi & \\ & n^{-1}\xi \end{pmatrix}\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t-x+u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} k_h\right)\psi(\alpha t)\, dt$$

$$= \psi(\alpha(x-u))\,\omega(\xi^{-1}n)\,|y|\,\hat{F}_{k_g,k_h,y^{-1}v}(\alpha y) \qquad \left[t \to t+x-u, \text{ then } t \to yt\right]$$

$$F_{k_g,k_h,y}(t) = \varphi_\omega\left(k_g^{-1}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k_h\right)$$

$$\boxed{\hat{F}_{k_g,k_h,y^{-1}v}(\alpha y) = \int_A \varphi_\omega\left(k_g^{-1}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{-1}v & 0 \\ 0 & 1 \end{pmatrix} k_h\right)\psi(\alpha y t)\, dt}$$

$\cdot K \, supp(\phi) K \subset GL(2, \mathbb{A})$ = continuous image of the compact set $K \times Supp(\phi) \times K$ and hence compact.

there exists a compact subset Ω of $\mathbb{A}^\times$

such that $F_{kg, kh, y}(t) \neq 0$, for any t, then $y \in \Omega$.

$$\left| K'(g,h) \right| \leq |y| \sum_{\alpha \in F^\times} \underbrace{\hat{F}_{kg, kh, g^{1}u}(\alpha y)}$$

Schwartz function of y, with support lies in $K \times K \times \Omega$.

(Paley-Wiener theorem)

Therefore if $N > 0$ is given, there exists a constant C_N such that, $|K'(g,h)| \leq C_N |y|^{-N}$.

Thus $\left| \rho(y) f(g) \right| \leq C_N |y|^{-N} \| f \|_1 \leq C C_N |y|^{-N} \|f\|_2$

L^1 norm defined over.

$F \backslash \mathbb{A}^\times \times \Omega \times K$

finite measure.

$\Leftarrow$

Therefore

$\|f\|_1 \leq C \|f\|_2$

Hence (i) is proved.

(ii) To show $\rho(y)$ is compact one applies Arzela-Ascoli. It suffices to show image of unit ball in $L^2_{cusp}(\cdot\,\cdot)$ is equicontinuous. Now $\exists$, $K \subseteq GL(2, \mathbb{A}_f)$ open compact such that y is right invariant under K. Therefore $\rho(y) f$ is also right invariant under K. Hence it is enough to

Show that $(\rho(\varphi) f)(g)$ are equicontinuous (2)

as functions of the infinite component g_∞.

Let Σ is the image of the unit ball in $L^2_{cusp}(\cdots)$.

we extend each $\rho(\varphi) f$ to the compactification of $\frac{GL(n,A)}{GL(n,F)Z(A)}$ by making it vanishing on the cusps. From (i) ~~we get that~~ Σ is bounded on L^∞ norm. To show Σ is equicontinuous it is enough to show the derivatives are bounded uniformly for all f with $\|f\|_2 < 1$.

$$(X\rho(\varphi) f)(g) = \frac{d}{dt} \rho(\varphi) f(ge^{tX})\Big|_{t=0}$$

$$= \rho(\varphi_X) f(g).$$

where $\varphi_X(g) = \frac{d}{dt}\varphi(e^{-tX}g)\Big|_{t=0}$.

Hence whatever estimate we have applied for $\rho(\varphi)$ also can be applied for $\rho(\varphi_X)$, hence it's derivatives.

The family Σ is therefore equicontinuous, and Σ is therefore compact in L^∞-norm hence in L^2-norm.

(As the measure space is finite)

Another way to see the compactness :→

Trace class operators are Hilbert-Schmidt and Hilbert-Schmidt are compact.

Dixmier-Malliavin lemma $\Rightarrow$ $f = \sum_{k=1}^{n} c_k f_1^k * f_2^k$

Hence $\rho(f) \Rightarrow$ finite linear combination of convolutions of two Hilbert-Schmidt, hence trace class.

$\underline{Th^m}$:→ The space of $L^2_{cusp}(\omega)$ forms decomposes into discrete ② Hilbert direct sum with finite multiplicities of irreducible unitary sub-representations.

$$L^2_{cusp}(\omega) = \bigoplus m(\pi) V_\pi, \quad m_\pi < \infty$$

<u>Connection with classical form</u>

Let f be cusp form for f for $SL_2(\mathbb{Z})$ of weight m.

$$f \longrightarrow \varphi \longrightarrow \underbrace{(\pi_\varphi, V_\varphi)}$$

→ admissible subspace of the space of cuspidal automorphic forms. Now if f is simultaneous eigenfunctions of all the classical Hecke-operator, then (π_φ, V_φ) is irreducible cuspidal representations.

Therefore we can apply the decomposition theorem to get $\pi_\varphi = \pi_\infty \otimes \left(\otimes' \pi_p\right)$ where

i) π_∞ is completely determined by the weight m of f.

ii) π_p is completely determined by the Hecke-eigenvalue $\lambda(p)$ of T_p acting on f.