Existence of Cusp forms for SL(2)

$\Delta = -y^2(\partial_{xx} + \partial_{yy}) \rightarrow$ Hyperbolic Laplacian on $\mathbb{H}$.

$L^2(PGL(2,\mathbb{Z})\backslash\mathbb{H}) = \langle 1 \rangle \oplus L^2_{cts} \oplus L^2_{cusp}$

L^2_{cts} ↓ generated by $E_{1/2+ir}(\gamma, z) = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \operatorname{Im}(\gamma z)^{1/2+ir}$

$L^2(PGL(2,\mathbb{Z})\backslash\mathbb{H})$ ↓ Spherical L^2-automorphic form.

We wish to show $L^2_{cusp} \neq 0$. We also have the Hecke operator T_p, for any prime p, which acts on functions in $L^2(PGL(2,\mathbb{Z})\backslash\mathbb{H})$, via the following rule:

$$T_p f(z) = \frac{1}{\sqrt{p}}\left(f(pz) + \sum_{k=0}^{p-1} f\left(\frac{z+k}{p}\right)\right)$$

[Def^n: $(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ 0 \le b \le d}} f\left(\frac{az+b}{d}\right)$]

Now we have the following action of Δ & T_p on the Eisenstein Series:

[on Eisenstein Series $E_s(z)$,
$T_n E_s(z) = \sum_{ad=n} \left(\frac{a}{d}\right)^s = \sum_{d|n} d^{1-2s}$]

- $\Delta E_{1/2+ir}(z) = \left(\frac{1}{2}+ir\right)\left(1-\frac{1}{2}-ir\right) E_{1/2+ir}(z)$
$= \left(\frac{1}{4}+r^2\right) E_{1/2+ir}(z)$.

- $T_p E_{1/2+ir}(z) = (p^{ir} + p^{-ir}) E_{1/2+ir}$ (After correct normalization)

Δ, & T_p commute with each other

The above two equation has a nice interpretation in terms of automorphic wave equation:

$$\Delta_{tt} = -\Delta u + u/4 ,$$

Let us assume the solution to the above equation is $u(x+iy, t)$ when we have the following boundary condition

$$u|_{t=0} = E_{1/2+ir}, \quad u_t|_{t=0} = 0 .$$

Then we get $u(x+iy, t) = \frac{1}{2} E_{1/2+ir}(x+iy)\left(e^{itr} + e^{-itr}\right)$

$$\left[\text{ as } \Delta_{tt}\left(u|_{t=0}\right) = -r^2 \left(u|_{t=0}\right) \right]$$

Hence $T_p E_{1/2+ir}(x+iy) = 2u(x+iy, \log p)$

Now let $U_t \in \mathrm{End}_{\mathbb{C}}\left(L^2(PGL(2,\mathbb{Z})\backslash\mathbb{H}) \cap C^\infty(PGL(2,\mathbb{Z})\backslash\mathbb{H})\right)$

$$f(x+iy) \longmapsto 2u(x+iy, t)$$

when u is the solution to $\Delta_{tt} = -\Delta u + u/4$, with boundary condition $u|_{t=0} = f$, $u_t|_{t=0} = 0$. Moreover one can show that U_t is self adjoint.

formally one can write $U_t = e^{t\sqrt{1/4 - \Delta}} + e^{-t\sqrt{1/4+\Delta}}$.

Therefore we have the following proposition

Prop For every $f \in L^2_{cts} \oplus \langle 1 \rangle$. both T_p & $U_{\log p}$ are self adjoint.

$$T_p f = U_{\log p} f ,$$

Cor for every smooth $f \in L^2(PGL(2,\mathbb{Z})\backslash\mathbb{H})$, $[T_p - U_{\log p}] f \in L^2_{cusp}$.

Now in order to Show ~~$T_p = V_{\log p}$~~ $)$. (25)

$L^2_{cusp}(\cdot) \neq 0$, it is enough to Show that there exists one non zero function f, such that

$$\left[T_p - V_{\log p}\right](f) \neq 0 .$$

For $R>0$, let Ω_R be the Sigel domain ~~$\begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix} \backslash \{x+iy, y>R\}$~~ .

we get the follow composition.

Now for $R>1$, we get the followng composition,

$$\Omega_R \hookrightarrow SL(2,\mathbb{Z}) \backslash \mathbb{H} \longrightarrow \frac{\mathbb{H}}{\left(SL(2,\mathbb{Z}), A : \det(A) = -1\right)}$$

$$= \frac{\mathbb{H}}{PGL(2,\mathbb{Z})}$$

Hence the image of Ω_R in $\frac{\mathbb{H}}{PGL(2,\mathbb{Z})}$ is quotienting out by $x+iy \longmapsto -x+iy$. Let $L^2_{0,even}(\Omega_R)$ be the set of function $\subset L^2(\Omega_R)$ with integral 0 and invariant under $x+iy \longrightarrow -x+iy$.

Hence we have isometric embedding.

$$L^2_{0,even}(\Omega_R) \cong L^2_0\left(PGL(2,\mathbb{Z}) \backslash \mathbb{H}\right).$$

Similarly $C^\infty_{0,even}(\Omega_R) \to$ Smooth subspace of $L^2_{0,even}(\Omega_R)$.

Let $V_{n,R}$ Subspace $\subset C^\infty_{0,even}(\Omega_R)$ consisting of functions of the form. $f(x+iy) = h(y)\cos(2\pi n x)$.

Let $R > e^{t}$. The solutions to the wave equation propagate @ a finite speed. It follows that if $f \in V_{n,R}$ has support on Ω_R, then the function $U_t f$ has support on t neighbourhood of Ω_R, i.e. on $\Omega_{Re^{-t}}$ (this is due to the metric is in hyperbolic space $ds^2 = y^{-2}(dx^2+dy^2)$. Here $d((x,y),(x,R)) = \left|\int_y^R \frac{dy'}{y'}\right| = \left|\ln \frac{R}{y}\right|$).

Now $\mathbb{R}/\mathbb{Z}$ acts naturally on $\Omega_{Re^{-t}}$, through the following $s \cdot f(x+iy) = f(x+s+iy)$. This action commutes with Δ, hence with U_t. Now a function $f \in C^\infty_{0,\text{even}}(\Omega)$ belongs to $V_{n,R}$ exactly when it transforms under under the action of $\mathbb{R}/\mathbb{Z}$ some linear combinations of characters of the form $s \mapsto e^{2\pi i n s}$ and $s \mapsto e^{-2\pi i n s}$ we have that

So when $R > e^{-t}$

$$U_t V_{n,R} \subset V_{n,Re^{-t}}.$$

On the other hand

$$T_p V_{n,R} \subset \begin{cases} V_{pn,R/p}, & \text{if } p \nmid n \\ V_{pn,R/p} \oplus V_{n/p,pR} & \text{if } p \mid n. \end{cases}$$

Now $V_{n,R}$ and $V_{n',R'}$ are orthogonal, when $n \neq n'$.

if $n \neq 0$ and $R > p$.

Therefore $T_p V_{n,R} \perp U_{\log p} V_{n,R}$

So if $f \in V_{n,R}$ and $T_p f \neq 0$, then $[T_p - U_{\log p}] f \neq 0$.

Now one can understand the operators T_p and $\mathcal{U}_{\log p}$ as a functions in the convolutions algebra of compactly supported smooth bi - $PO(\mathbb{R})$ invariant functions / distribution.

$$L^2\left(PGL(2,\mathbb{Z})\backslash PGL(2,\mathbb{R})/PO(\mathbb{R})\right) \simeq L^2\left(PGL(2,\mathbb{Z})\backslash \mathbb{H}\right).$$

Therefore the functions on the right are $K_\infty = PO(\mathbb{R})$ invariant functions on $PGL(2,\mathbb{Z})\backslash PGL(2,\mathbb{R})$

Let $\varphi \in C_c^\infty(k\backslash G/k)$ or distributions. then $f * \check{\varphi}(g) = \int_G \varphi(gh^{-1}) f(h)\,dh$.

Now, let $\Sigma_s(z)$ be the Harish-chandra Spherical function defined on $\mathbb{H}$ which is characterized by

Unique characterization upto a Scalar:

i) Spherically symmetric, i.e. depending only on the hyperbolic distance of z from the point $i \in \mathbb{H}$.

ii) $\Delta \Sigma_s = \left(\tfrac{1}{4}+s^2\right)\Sigma_s(z)$

iii) $\Sigma_s(i) = 1$.

Now for any φ as above (as a distribution & bi k_∞ invariant), satisfy the following : $\Sigma_s * \check{\varphi}$ also satisfies i) & ii).

Hence $\Sigma_s * \check{\varphi} = \hat{\varphi}(s)\,\Sigma_s$, when $\hat{\varphi}(s)$ is called the Harish-chandra Spherical fourier transform.

Now the inverse of this

$$\varphi(z) = \frac{1}{4\pi^2}\int_{\mathbb{R}} \hat{\varphi}(s)\,\Sigma_s(z)\,|c(s)|^{-2}ds \qquad c(s) = \pi^{-1/2}\frac{\Gamma(is)}{\Gamma(is+\tfrac{1}{2})}$$

($|c(s)|^{-2}ds$: plancherel measure.)

Therefore, by definition of $\mathcal{U}_{\log p}$, it's spectral transform is $p^{is} + p^{-is}$, Hence formally the corresponding distribution (spherical) is given by

$$K_{\mathcal{U}_{\log p}}(z) = \frac{1}{4\pi^2}\int \left(p^{is} + p^{-is}\right) \Sigma_s(z) \left[c(s)\right]^{-2} ds.$$

it has a support in the hyperbolic disc of radius $\log p$ around the point $i \in \mathbb{H}$ (using Paley-Wiener theorem).

<u>Now for T_p we do the following:</u>

When $S = \{\infty, p\}$, then, $PGL(2,\mathbb{Z}[1/p]) \backslash PGL(2,\mathbb{R}) \times PGL(2,\mathbb{Q}_p) / K_\infty \times PGL(2,\mathbb{Z}_p)$

$$\cong PGL(2,\mathbb{Z}) \backslash PGL(2,\mathbb{R}) / K_\infty.$$

we denote this isomorphism as $\hat{i}$.

then we have isomorphism $i^* : L^2(PGL(2,\mathbb{Z}) \backslash \mathbb{H}) \to L^2(\Gamma \backslash G_S / K_S)$.

Therefore we have $i^*(T_p f) = i^*(f) * K_{T_p}$.

When $K_{T_p}(g_\infty, g_p) = p^{-1/2} \delta_1(g_\infty) \mathbb{1}_{K_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_p}(g_p)$

Therefore the operator $\mathcal{N} = T_p - \mathcal{U}_{\log p}$ on $L^2(PGL(2,\mathbb{Z}) \backslash \mathbb{H})$ can be viewed as convolution operator on the space $L^2(\Gamma \backslash G_S / K_S)$ by the distribution $K_{\mathcal{N}} = K_{\mathcal{U}_{\log p}} - K_{T_p}$.

and for any $H = H_\infty \mathbb{1}_{K_\infty}$ $(\in C_c^\infty(K_S \backslash G_S / K_S))$ the composition $f \to (\mathcal{N} f) * H$ give rise to following convolution operator $f \to f * (k_1 - k_1')$. using cuspidality of the image we get necessary condition on $\widehat{k_1 - k_1'}$, then we find $k_1 - k_1'$ using paley wiener.