Recap (Points P.O.V): Let X be an scheme over a field F.
$\forall$ F-algebra R, we consider
$$X(R) := \mathrm{Hom}_F(F[X], R) = R\text{-points of } X \ (= \text{a set}).$$
"slogan": the S-points of X, for every S, characterize X.

Analogy Blueprint of a House : scheme, Houses : R-points, Generic House : F-points

When R is a locally compact ring, we endow $G(R)$ with a locally compact topology.

Actions and Orbits: Let X be a scheme over a ring Field F and H be a smooth algebraic group over F.

Def An action is $a: H \times X \to X$ such that

- $$\begin{tikzcd} X \times H \times H \arrow[r, "a \times 1"] \arrow[d, "a \times m"'] & X \times H \arrow[d, "a"] \\ X \times H \arrow[r, "a"] & X \end{tikzcd}$$ commutes

- $X \xrightarrow{1 \times e} X \times H \xrightarrow{a} X$ is the identity where $e: \mathrm{Spec}(F) \to H$ is the identity.

Rmk

- $H(R)$ is a group for every F-algebra R.
- H-action of X is the same as an $H(S)$-action for every S

For $\gamma \in H(F)$, we consider $a(\gamma, \cdot): H \to X$. Also consider

$$\begin{tikzcd} H_\gamma = \mathrm{Spec}(F) \times_X H \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d, "a(\gamma,\cdot)"] \\ \mathrm{Spec}(F) \arrow[r, "\gamma"'] & X \end{tikzcd}$$

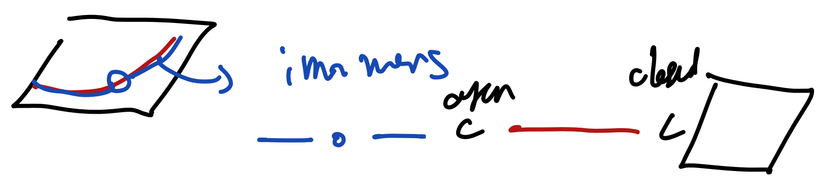
For F-algebras R,

$$H_\gamma(R) = \left\{ h \in H(R) : \begin{tikzcd} \mathrm{Spec}(R) \arrow[rr, "a(\gamma,h)"] \arrow[dr] & & X \\ & \mathrm{Spec}(F) \arrow[ur, "\gamma"'] & \end{tikzcd} \right\}$$

i.e. "$a(\gamma, h) = \gamma$"

Observe that $H_\gamma \subset H$ is closed subscheme

Prop: $H_\gamma \backslash H \longrightarrow X$ is an immersion = closed immersion and then open



Rmk. ② Quotients: $X \to Y$ surjective $\not\Rightarrow$ $X(F) \to Y(F)$ surjective

But if $F = \bar{F}$, it is true.

In particular $\left(H_\alpha \backslash H\right)(F) \neq H_\alpha(F) \backslash H(F)$

Prop Let I be an affine smooth alg. subgroup of H.

Then

$$\left(I \backslash H\right)(F) = \{ I(F^{sep}) h \in I(F^{sep}) \backslash H(F^{sep}) : h \cdot \xi(h^{-1}) \in I(F^{sep}) \ \forall \xi \in Gal_F \}$$