(Global) Let F be a global field. Let us fix $\chi : H(\mathbb{A}_F) \to \mathbb{C}^\times$ character trivial on $H(F)$. Let S' be a finite subset of places.

Restricted product: we fix a model of X over $\mathcal{O}_F^S$.

- $C_c^\infty(X(\mathbb{A}_F^{S'})) = \bigotimes'_{\substack{v \notin S' \\ v \nmid \infty}} \underbrace{C_c^\infty(X(F_v))}_{\text{locally constant}} \otimes \bigotimes_{\substack{v \notin S' \\ v \mid \infty}} \underbrace{C_c^\infty(X(F_v))}_{\text{smooth}}$
- $\gamma \in X(F)$ is χ-relevant if χ is trivial on $H_\gamma(\mathbb{A}_F)$
- γ is relatively unimodular if $\delta_H|_{H_\gamma} = \delta_{H_\gamma}$

Global relative orbital integral

$$\boxed{RO_\gamma^\chi(f^{S'}) = \int_{H_\gamma(\mathbb{A}_F^{S'}) \backslash H(\mathbb{A}_F^{S'})} f^{S'}(\gamma \cdot h)\, \chi(h)\, \frac{d_r h}{d_r h_\gamma}.}$$

Prop If $\gamma \in X(F)$ is relevant, relatively semi simple, relatively unimodular and f is smooth then $RO_\gamma(f^s)$ is absolutely convergent.

The idea again is to prove that

$$H_\gamma(\mathbb{A}_F) \backslash H(\mathbb{A}_F) \longrightarrow X(\mathbb{A}_F) \text{ is proper.}$$

↳ Pre image of compact is compact.

Origami We want to study

$$\int_{H(F)\backslash H(\mathbb{A}_F)} \sum_{\gamma \in X(F)} f(\gamma \cdot h)\, \chi(h)\, dh.$$

This integral has very delicate convergence. But let us manipulate formally:

$$\int_{H(F)\backslash H(A_F)} \sum_{\gamma} f(\gamma \cdot h) \cdot \chi(h)\, d_r(h) = \int_{H(F)\backslash H(A_F)} \sum_{[\gamma]} \sum_{\gamma' \in [\gamma]} f(\gamma' \cdot h) \quad \chi(h)\, dh.$$

$$X(F) = \bigcup_{[\gamma]} \gamma \cdot H(F)$$

$$\gamma \cdot H(F) \longleftrightarrow H_\gamma(F)\backslash H(F)$$

$$= \sum_{[\gamma]} \int_{H_\gamma(F)\backslash H(A_F)} f(\gamma \cdot h)\chi(h)\, dh$$

$$H(A_F) \supset H_\gamma(A_F) \supset H_\gamma(F)$$

$$= \sum_{[\gamma]} \int_{H_\gamma(F)\backslash H_\gamma(A_F)} \chi(h_\gamma)\, dh_\gamma \int_{H_\gamma(A_F)\backslash H(A_F)} f(\gamma \cdot h)\chi(h)\, \frac{d_r h}{d_r h_\gamma}$$

$$(\gamma \ \chi\text{-relevant} \ldots) \quad = \sum_{[\gamma]} \mathrm{Vol}_{d_{h_\gamma}}\left(H_\gamma(F)\backslash H_\gamma(A_F)\right) \underbrace{}_{<\infty} RO_\gamma^{\chi}(f)$$

$$(\gamma \text{ relative unimodular}) \quad = \sum_{[\gamma]} \tau(H_\gamma)\, RO_\gamma^{\cancel{\chi}}(f)$$

γ is relative elliptic.

Convergence: As we saw we need to adapt a bit our goal: let H^0 be the connected component of H, and assume that $H^0 = H_u \times H_r$ where H_u is a unipotent subgroup (e.g. $\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$) and H_r is a reductive one. For smooth connected

Rmk In general we have these decomposition but it is a semi direct product.

Let C_X be the stabiliser of X under H_r, and let

$$A := A_{C_X} = \text{"split central component" of } Z(C_X).$$

e.g. $F = \mathbb{Q}$

$$A_{GL_2} = \mathbb{R}_{>0} \mathrm{Id} \subset GL_2(\mathbb{R}) \subset GL_2(\mathbb{A}_\mathbb{Q}).$$