Proof: Idea We assume that $H = H^0$ [H/H^0 is finite].

One can show that (the adjectives are transported via the H-action.)

$$\int_{A_H(F)\backslash H(\mathbb{A}_F)} \sum_{\gamma} |f(\gamma \cdot h)\, \chi(h)|\, dh = \sum_{\mathcal{O}} \underbrace{\int_{A_H(F)\backslash H(\mathbb{A}_F)} \sum_{\gamma \in \mathcal{O}(F)} |f(\gamma \cdot h)\, \chi(h)|\, dh.}_{(*)}$$

where the sum is over closed H-orbits in X, s.t. $\forall \gamma \in \mathcal{O}(F)$ has the closed orbit and is relatively unimodular and relatively elliptic.

$\mathcal{O} \simeq H/H_\gamma$

$$(*) = \sum_{\substack{\gamma \\ H(F)\text{-orbit} \\ \text{in } \mathcal{O}(F)}} \int_{A_{H_\gamma}(F)\backslash H(\mathbb{A}_F)} |f(\gamma \cdot h)\chi(h)|\, dh$$

$$= \sum_{\gamma} \underbrace{\tau(H_\gamma)}_{< \infty}\, RO^{\chi}_{\gamma}(f)$$

(Finite Sums)

Inner: Under the gcf condition (more precisely $H^1 < \infty$).
Let Ω be a compact subset of $\mathcal{O}(\gamma)(\mathbb{A}_F^\infty)$. Then are only finitely many $H(F)$-orbit $xH(F) \subset X(F)$ s.t. $xH(\mathbb{A}_F^\infty) \cap \Omega \neq \phi$. (Theorem 17.6.10)

Proof Omitted ... Galois cohomology.

(Lemma 17.6.11)

Outer: Let $\Omega \subset X(\mathbb{A}_F)$ be a compact subset. There are only finitely many closed H_r-orbit $\mathcal{O} \in (X/H)(F)$ such that $\gamma H_r(\mathbb{A}_F) \cap \Omega \neq \phi$ for some $\gamma \in \mathcal{O}(F)$.

Proof. {space of orbits} $\leftrightarrow$ G.I.T quotient (X/H_r)

$\leftrightarrow$ Affine scheme

H_r is reductive

$\cdot \{ \mathcal{O} \in X/H_r(F) : \gamma H(\mathbb{A}_F) \cap \Omega \neq \phi$, for some $\gamma \in G(F) \}$.

$\Updownarrow$

$\mathcal{O}$ is a closed H_r-orbit.

$\cap$

$$P(\Omega) \cap [X/H_r](F)$$

where $p : X(\mathbb{A}_F) \longrightarrow (X/H_r)(\mathbb{A}_F)$.

Since X/H_r is affine

$(X/H_r)(F) \hookrightarrow (X/H_r)(\mathbb{A}_F)$ discrete and closed!

and $P(\Omega)$ compact, we can conclude.

Simple Relative trace formula.

F = number field.

Consider $\underset{G\times G}{\overset{\cap}{H}} \curvearrowright \underset{G}{\overset{\shortparallel}{X}}$; $g(g_1,g_2) = g_1^{-1} g g_2$

Let

- A_G = "split component of $Z(G)$" $\subset Z_G(F \otimes_{\mathbb{Q}} \mathbb{R})$.
- $A_{G,H} = H(\mathbb{A}_F) \cap (A_G \times A_G)$.
- $\chi : A_{G,H} H(F) \backslash H(\mathbb{A}_F) \to \mathbb{C}^\times$

(relative traces) Recall the spectral decomposition of allows us to $K_{f,\mathrm{cusp}}$ (= kernel $R(f)$ on $L^2_{\mathrm{cusp}}(A_G \backslash G(\mathbb{A}_F))$; $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ as $\sum_{\pi \subset L^2_{\mathrm{cusp}}} K_{\pi(f)}$. Let $\pi \subset L^2_{\mathrm{cusp}}$, we define

$$\boxed{\mathrm{rtr}_{H,\chi}\,\pi(f) = \int_{A_{G,H} H(F) \backslash H(\mathbb{A}_F)} K_{\pi(f)}(h_\ell, h_r)\, \chi(h_\ell, h_r)\, d(h_\ell, h_r) \quad \forall f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))}$$

(orbital integral bis.) Assume as usual that γ is relevant, relatively semisimple and relatively unimodular

$$\boxed{RO_\gamma(f) := \int_{A \backslash H_\gamma(A_F) \backslash A_{G,H} \backslash H(A_F)} f(h_1^{-1}\gamma h_2)\, \chi(h_1,h_2)\, \frac{d(h_1,h_2)}{dh_\gamma}}$$

Rmk
- $A = H(A_F) \cap \Delta(A_G)$. (lemma 18.2.2)
- If $A = A_{G,H}$; then RO_γ is the usual orbital integral.

Thm Let $f \in C_c^\infty(A_G \backslash G(A_F))$ such that
- $\mathrm{im}(R(f)) \subset L^2_{cusp}$.
- $H(A_F)$-orbit of γ intersects the support of f, then γ is relatively elliptic, relatively unimodular, and the H_γ-orbit of γ is closed.

Then

$$\boxed{\sum_\pi rtr\, \pi(f) = \sum_{[\gamma]} \tau(H_\gamma)\, RO_\gamma(f).}$$

Rmk Let π be a cuspidal representation of $A_G \backslash G(\mathbb{A}_F)$.

Then $\operatorname{tr} \pi(f) \neq 0$ for some f iff $\pi \otimes \pi^\vee$ is (H, χ)-distinguished.

$$\hookrightarrow \mathcal{P}_\chi(\varphi) = \int_{A_{G} \cap H(F) \backslash H(\mathbb{A}_F)} \varphi(h) \chi(h) \, dh \neq 0$$

$\exists \varphi \in L^2(\pi \otimes \pi^\vee)$ smooth.