· F: number field · $H \le G\times G$, $X = G$. – $H \curvearrowright X$ | $\chi: A_{G,H}\backslash H(\mathbb{A}_F) \to \mathbb{C}^\times$

· $A_G :=$ "Split component of Z_G" $\subseteq Z_G(F\otimes_{\mathbb{Q}}\mathbb{R})$

$R(f)(L^2([G])) \subseteq L^2_{cusp}([G])$

Theorem (Simple relative trace formula)

$f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ s.t. has cuspidal image

① $R(f)$ [def. by $(R(f)\phi)(g) := \int_{A_G\backslash G(\mathbb{A}_F)} f(x)\phi(gx)\,dx$)

② if $H(\mathbb{A}_F)$-orbit of γ intersect with the support of f

$\Rightarrow$ γ relatively elliptic, relatively unimodular, & H_γ-orbit of γ is closed.

$$\Rightarrow \sum_{\pi} \mathrm{rtr}\, \pi(f) = \sum_{[\gamma]} \tau(H_\gamma)\, RO_\gamma(f)$$

(H,χ)-distinguished cuspidal auto. rep — relevant relative classes: rel. unimodular, rel. elliptic s.t. H_γ is closed.

$$\mathrm{rtr}\,\pi(f) := \int_{A_{G,H}H(F)\backslash H(\mathbb{A}_F)} K_{\pi(f)}(h_1,h_2)\,\chi(h_1,h_2)\,d(h_1,h_2)$$

(Note that $H \subseteq G\times G$.)

다음.. 정리에서 증명할 거 시도..

· F: number field. · G: reductive group over F.
$H \le G\times G$ & ~~X~~ $H \curvearrowright X \cong G$. (0 나중!)

$\Rightarrow \exists A_G$,

Def · $A_{G,H} := H(\mathbb{A}_F) \cap A_G \times A_G$

· $\chi: A_{G,H}\backslash H(\mathbb{A}_F) \to \mathbb{C}^\times$. a character.

· $[G] := A_G G(F)\backslash G(\mathbb{A}_F)$ $\quad f \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ "test function"

· $R(f): L^2([G]) \to L^2([G])$ integrated operator

$$\phi \longmapsto \left[g \longmapsto \int_{A_G\backslash G(\mathbb{A}_F)} f(x)\phi(gx)\,dx\right]$$

대한 adelic point.

Lemma 6.1.2 · if $R(f)(L^2([G])) \subseteq L^2_{cusp}([G])$, then it's "Cuspidal images".

"π-component" · $K_f(x,y) := \sum_{\gamma\in G(F)} f(x^{-1}\gamma y)$, kernel ftn $\Leftrightarrow R(f)\phi(x) = \int_{[G]} K_f(x,y)\phi(y)\,dy$

Since $A_G\backslash G(\mathbb{A}_F) \curvearrowright L^2_{cusp}([G])$ by right multiplication, $(g.\phi)(x) := \phi(xg)$

& as $A_G\backslash G(\mathbb{A}_F)$-representation, [Gr 9.1.2]

$$L^2_{cusp}([G]) \cong \widehat{\bigoplus_\pi} L^2_{cusp}(\pi)$$

We define

$$\pi(f) := R(f)\big|_{L^2_{cusp}(\pi)}$$

We want to define $K_{\pi(f)}(x,y)$, s.t. $K_{f,cusp}(x,y) = \sum_\pi K_{\pi(f)}(x,y)$

$(x,y) \in [G]$ $\quad K_{\pi(f)} \in L^2([G]\times[G])$

Why? $\psi = \sum_{\phi \in B_\pi} \langle \psi, \phi \rangle \phi$

$\Rightarrow \psi(x) = \sum_{\phi \in B_\pi} \phi(x) \int_{[G]} \psi(y) \overline{\phi(y)} dy \xrightarrow[\text{dense}]{} K_{\Sigma,\pi}(x,y) = \sum_{\phi \in B_\pi} \phi(x) \overline{\phi(y)}$.

$\pi(f)\psi = \sum_{\phi \in B_\pi} \langle \psi, \phi \rangle \pi(f)\phi$

For a specific basis B_π,

$$K_{\pi(f)}(x,y) := \sum_{\phi \in B_\pi} (\pi(f)\phi)(x) \cdot \overline{\phi(y)}$$

will be a kernel function. (Theorem 16.2.3)

Using this, we obtain ordinary trace (16.9)

$$\sum_\pi m(\pi) \underbrace{\mathrm{tr}(\pi(f))}_{\sum_{\phi \in B_\pi} \langle \pi(f)\phi, \phi \rangle} = \sum_\pi \int_{[G]} K_{\pi(f)}(x,x)\, dx$$

Relative means "admitting non-diagonal"

$H \leq G \times G$, $X = G$. $H \curvearrowright X$.

$\exists A_G \leq Z_G(F \otimes_{\mathbb{Q}} \mathbb{R})$. $A_{G,H} := H(\mathbb{A}_F) \cap A_G \times A_G$ "admitting non-diagonal"

$[G] = A_G G(F) \backslash G(\mathbb{A}_F)$ 여기 $[G] \times [G]$ 생각 ...

$\chi : A_{G,H} \backslash H(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times$ character.

We can write $\chi(h)$, or $\chi(h_1, h_2)$.

$[G] = A_G G(F) \backslash G(\mathbb{A}_F) \dashrightarrow L^2([G] \times [G])$

$A_{G,H} H(F) \backslash H(\mathbb{A}_F)$ 로 바뀌는 것이 자연스럽다...

(Definition of $K_{\pi(f)}$ do not need to be diag. $[G] \times [G]$)

Def $\text{rtr}\, \pi(f) := \int_{A_{G,H} H(F) \backslash H(\mathbb{A}_F)} K_{\pi(f)}(h_1, h_2) \cdot \chi(h_1, h_2)\, dh$.

(즉 여기는 kernel 함수를 적분하는 것)

$H \curvearrowright X$.

$H \leq G \times G$, $H \curvearrowright X$

Σ_n §17 (Hecke...)

$\gamma \in X(F)$ $f \in C_c^\infty(X(F))$

$H_\gamma = H \times_\gamma \mathrm{Spec}\, F$ "Gal of stabilizer" Subscheme

$$RO_\gamma^\chi(f) := \int_{H_\gamma(F) \backslash H(F)} f(\gamma h) \chi(h) \frac{dh}{dh_\gamma}.$$

$$(f^{S'}) := \int_{H_\gamma(\mathbb{A}_F^{S'}) \backslash H(\mathbb{A}_F^{S'})} \cdots \frac{dh}{dh_\gamma \cdots}$$

Def $RO_\gamma(f) := \int_{(A \backslash H_\gamma(\mathbb{A}_F)) \backslash (A_{G,H} \backslash H(\mathbb{A}_F))} f(h_1^{-1} \gamma h_2) \cdot \chi(h_1, h_2) \frac{dh}{dh_\gamma}$

Only difference comes from the range.

$H_\gamma(\mathbb{A}_F^{S'}) \backslash H(\mathbb{A}_F^{S'})$ v.s. $(A \backslash H_\gamma) \backslash (A_{G,H} \backslash H)$

$f \in C_c^\infty(X(F))$ $\qquad f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$

$A = H(\mathbb{A}_F) \cap A_G \times A_G$

$H \to A_{G,H} \backslash H$

$H_\gamma \to A \backslash H_\gamma$ ... even non-relative ordinary trace. We used $[G] = A_G G(F) \backslash G(\mathbb{A}_F)$

Theorem (18.2.4) $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ s.t.

1) $R(f)(L^2[G]) \leq L^2_{cusp}[G]$

2) If $H(\mathbb{A}_F)$-orbit of γ intersects with support of f
$\Rightarrow$ relatively elliptic, relative unimodular (well-def), H_γ-orbit is closed (convergence).

Then.
$$\sum_\pi \text{rtr}\, \pi(f) = \sum_{[\gamma]} \tau(H_\gamma) \cdot RO_\gamma(f)$$

(H, X)-dist. conj. classes

rel. unimodular
rel. elliptic.
H_γ-orbit is closed.