RO parts only use $G(\mathbb{A})/H(\mathbb{A}) \cong H_1(\mathbb{A})\backslash G(\mathbb{A})/H_2(\mathbb{A})$ $\blacksquare$

(Need to check unimodularity ....)

Note RO parts are not split.

$$\text{"}\int_{\dots} f(h_1^{-1}\gamma h_2)\,\chi(h_1 h_2)\,\frac{dh}{dh_\sigma}$$

§18.9 θ: finite order elt of $\mathrm{Aut}_F(G)$.

θ-conjugation: $G(R)\times G(R) \longrightarrow G(R)$

$$(x, g) \longmapsto g^{-1}\cdot x\cdot \theta(g)$$

Applying $H = G\times\theta(G)$, we will obtain the "Twisted ordinary trace formula".

I will not give a detail, but ... it detects "twist invariants form".

$$I_\theta : L^2([G]) \longrightarrow L^2([G])$$

$$\varphi \longmapsto (g \longmapsto \varphi(\theta^{-1}(g))) \quad \text{preserves } L^2_{cusp}([G])$$

$$\pi^\theta(g) := \pi(\theta(g)) \qquad (\pi : G(\mathbb{A}_F) \xrightarrow{\theta_F} G(\mathbb{A}_F) \to GL(V_\pi))$$

Claim (18.29) $\pi(f)\circ I_\theta : \underbrace{L^2_\pi(\pi)}_{\pi\text{-isotypic in } L^2([G])} \longrightarrow L^2(\pi^{\theta^{-1}})$

$$= ([I_\theta\circ R(\theta(g))]\varphi)(x)$$

$\varphi \in L^2(\pi) \Rightarrow \{R(g)\varphi : g\in G(\mathbb{A}_F)\} \cong V_\pi$.

$$R(g)(I_\theta\varphi)(x) = (I_\theta\varphi)(xg) = \varphi(\theta^{-1}(xg)) = \varphi(\theta^{-1}(x)\theta^{-1}(g))$$

$$= (I_\theta(R(\theta^{-1}(g))\varphi))(x)$$

$$\Rightarrow R(g)\,I_\theta = I_\theta\,R(\theta^{-1}(g))$$

$$\Rightarrow I_\theta(\underbrace{L^2(\pi)}_{V_\pi}) = I_\theta(\mathrm{Span}\{\varphi, R(g)\varphi, \dots\})$$

$$= \mathrm{Span}\{I_\theta\varphi, I_\theta R(g)\varphi, \dots\}$$

$$= \mathrm{Span}\{I_\theta\varphi, R(\theta^{-1}(g))I_\theta\varphi, \dots\}$$

$$= \mathrm{Span}\{R(\theta^{-1}(g))I_\theta\varphi : g\in G(\mathbb{A}_F)\}$$

$$\boxed{(I_\theta\varphi)(g) = \varphi(\theta^{-1}(g)),\quad \pi^\theta(g) = \pi(\theta(g))}$$

$$\overset{(\theta:\text{unit})}{=} \mathrm{Span}\{R(g)I_\theta\varphi : g\in G(\mathbb{A}_F)\}$$

$$\cong V_{\pi^{\theta^{-1}}}$$

$\Rightarrow I_\theta(L^2(\pi)) = L^2(\pi^{\theta^{-1}})$. Since ($\pi(f)v := \int_G f(g)\pi(g)v\,dg$ is from usual trace formula.) test fn

Def (Twisted trace)

$$\mathrm{Tr}(\pi(f)\circ I_\theta) = \int_{[G]} K_{\pi(f)}(g, \theta(g))\,dg.$$

$\Rightarrow$ $\mathrm{Tr}(\pi(f)\circ I_\theta) = 0$ unless $\pi \cong \pi^{\theta^{-1}}$ ($\Leftrightarrow \pi\cong\pi^\theta$)

So, it detects θ-invariant automorphic representations.

Cor 18.5.1 $\sum \mathrm{tr}(\pi(f)\circ I_\theta) = \sum_{[\gamma]} \tau(C_\gamma^\theta)\cdot TO_\gamma(f)$

$$TO_\gamma(f) := \int_{A_G^\theta C_\gamma^\theta(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(g^{-1}\gamma\theta(g))\,\frac{dg}{dg_\gamma}$$

$C_\gamma^\theta(\mathbb{A}_G) = \{g\in G(R) : g^{-1}\gamma\,\theta(g) = \gamma\}$, $A_G^\theta = \{a\in A_G : \theta(a) = a\}$

$A_G H(F) \backslash A_G H(\mathbb{A}_F) =$

[In §18.3, we consider $G = H_1 \times H_2 \le H \times H$]

Def $H \le G$, reduct over F.

$\varphi : G(F) \backslash G(\mathbb{A}_F) \to \mathbb{C}$ "smooth"

$\chi : H(\mathbb{A}_F) \to \mathbb{C}^\times$ trivial on $A_G \cap H(\mathbb{A}_F)$ $\leadsto$ $\chi : A_G \cap H(\mathbb{A}_F) \backslash H(\mathbb{A}_F) \to \mathbb{C}^\times$

period integral [defined when absolute converge]

$$P_\chi(\varphi) := \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} \varphi(h) \cdot \chi(h)\, dh.$$

(if smooth, it is ...) (Aha.. χ should be trivial on $A_G \cap H(\mathbb{A}_F)$. a requirement)

$G(F)/H(F)$: set of $H(F)$-orbits

If $H = H_1 \times H_2$, $H(F) \curvearrowright G(F)$ by.. ~~...~~

$\gamma \cdot (h_1, h_2) := h_1^{-1} x h_2$

(so that right action)

$\therefore \gamma_1, \gamma_2 \in G(F)$ are in the same $(H_1 \times H_2)(F)$-orbits

iff $\gamma_1 \cdot (h_1, h_2) = \gamma_2$ for some $(h_1, h_2) \in H_1(F) \times H_2(F)$

$\Leftrightarrow h_1^{-1} \gamma_1 h_2 = \gamma_2 \Leftrightarrow \gamma_2 \in H_1(F) \gamma_1 H_2(F)$

~~...~~

setting **Assume** $G(F)/H(F)$ (orbit) $= H_1(F) \backslash G(F) / H_2(F)$ (double coset)

setting **Assume** . Technical condition on H.

. $\chi = \chi_1 \otimes \chi_2$, where χ_i, character on $H_i(\mathbb{A}_F)$.

$K_\infty :=$ maximal compact subgroup of $G(F_\infty)$

$f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ is K_∞-finite

if $\mathrm{Span}_{\mathbb{C}} \{ R(k) f : k \in K_\infty \}$ is f.d.

$(R(k)f)(x) := f(xk)$.

π, cuspidal automorphic rep of $A_G \backslash G(\mathbb{A}_F)$

(4.4.3) $\Rightarrow L^2_{cusp}(\pi) \supseteq \{K_\infty\text{-finite}\}$ dense.

$\Rightarrow \exists$ Orthonormal basis $B(\pi)$ of $L^2_{cusp}(\pi)$ of K_∞-finite

Cor 18.31 $C_c^\infty(\cdots)$ f, K_∞-finite, $R(f)$, cuspidal image. $H(\mathbb{A}_F)$-orbit of γ intersects w/ $\mathrm{Supp}(f) \Rightarrow \gamma$, rel. elliptic & γ, closed

$$\Rightarrow \sum_\pi \sum_{\substack{\varphi \in B(\pi) \\ \varphi \text{ smooth}}}^{\text{only finite nonzero}} P_{\chi_1}(\pi(f)\varphi)\, \overline{P_{\chi_2}(\varphi)} = \sum_{[\gamma] \in H_1(F) \backslash G(F) / H_2(F)} \tau(H_\gamma)\, RO_\gamma(f)$$

proof) $H = H_1 \times H_2$, relative trace formula settings..

$B(\pi)$, K_∞-finite basis, f, K_∞-finite $\Rightarrow$ finitely many $B(\pi)$ spans f ~~...~~..

$$K_{\pi(f)}(x, y) := \sum_{\varphi \in B(\pi)} (\pi(f)\varphi)(x) \cdot \overline{\varphi(y)}$$
$x = h_1$, $y = h_2$.

$$\Rightarrow \mathrm{rtr}_{\pi,\chi}(f) = \int_{A_G H(F) \backslash H(\mathbb{A}_F)} \sum_{\varphi \in B(\pi)} (\pi(f)\varphi)(h_1)\, \overline{\varphi(h_2)}\, \chi_1(h_1)\, \chi_2(h_2)\, dh.$$

$$= \int_{A_G H_1(F) \backslash A_G H_1(\mathbb{A}_F)} (\pi(f)\varphi)(h_1) \cdot \chi(h_1)\, dh_1 \times \int_{A_G H_2(F) \backslash A_G H_2(\mathbb{A}_F)} \overline{\varphi(h_2)} \cdot \chi_2(h_2)\, dh_2 = P_{\chi_1}(\pi(f)) \cdot \overline{P_{\chi_2}(f)}$$

For $\gamma \in G(F)$,

$$G_\gamma(R) := \{ g \in G(R) : g\gamma g^{-1} = \gamma \}$$

$$O_\gamma(f) := \int_{G_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f(g^{-1}\gamma g) \cdot \frac{dg}{dg_\gamma}$$

We choose $\chi = 1$ & $H = \Delta(G) \leq G \times G$.

Then

$A_{G,H} = H(\mathbb{A}_F) \cap A_G \times A_G$, "$A_{G,\Delta(G)} = \Delta(A_G)$"

$$\mathrm{rtr}(\pi(f)) = \int_{\Delta(A_G)\Delta(G)(F) \backslash \Delta(G)(\mathbb{A}_F)} K_{\pi(f)}(g,g)\, dg.$$

$\Delta(G) \simeq G$

$$\simeq \int_{A_G G(F) \backslash G(\mathbb{A}_F)} K_{\pi(f)}(g,g)\, dg.$$

$A_{G,H} = \Delta(A_G)$

$H = \Delta(G)$

$$= m(\pi)\, \mathrm{Tr}(\pi(f))$$

since $H_\gamma = \{ (h_1,h_2) \in H : h_1^{-1}\gamma h_2 = \gamma \}$,

if $H = \Delta(G)$, then $H_\gamma = \{ (g,g) \in \Delta(G) : g^{-1}\gamma g = \gamma \}$

$A = A_{G,H} \cap H_\gamma(\mathbb{A}_F)$

$$= \Delta(G_\gamma).$$

$$\therefore \Delta(A_G) \cap H_\gamma(\mathbb{A}_F) \backslash H_\gamma(\mathbb{A}_F) \simeq A_G \cap G_\gamma(\mathbb{A}_F) \backslash G_\gamma(\mathbb{A}_F)$$

$$A_{G,H} \backslash H(\mathbb{A}_F) \simeq \Delta(A_G) \backslash \Delta(G)(\mathbb{A}_F) \simeq A_G \backslash G(\mathbb{A}_F)$$

$$\therefore (A_G \cap G_\gamma(\mathbb{A}_F) \backslash G_\gamma(\mathbb{A}_F)) \backslash (A_G \backslash G(\mathbb{A}_F)) \simeq G_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F)$$

So, $\tau(H_\gamma)\, RO_\gamma(f)$,

$$= \tau(\Delta(G_\gamma)) \cdot \int_{(A \backslash H_\gamma(\mathbb{A}_F)) \backslash (A_{G,H} \backslash H(\mathbb{A}_F))} f(h_1^{-1}\gamma h_2) \cdot \chi(h_1 h_2) \frac{d(h_1,h_2)}{dh_\gamma}$$

$$= \tau(G_\gamma) \cdot \int_{G_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f(g^{-1}\gamma g) \frac{dg}{dg_\gamma}$$

$$= \tau(G_\gamma) \cdot O_\gamma(f)$$

All conditions may be relevant. $\square$

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Recall Relative trace formula + $H = \Delta(G) \Rightarrow$ ordinary trace formula.

Relative $\sum_{\pi} \mathrm{rtr}_{H,\chi}\, \pi(f) = \sum_{[\gamma]} \tau(H_\gamma)\, RO_\gamma(f)$ orbit space

$$\mathrm{rtr}_{H,\chi}(f) := \int_{A_{G,H} H(F) \backslash H(\mathbb{A}_F)} K_\pi(f)(h_1,h_2)\, \chi(h_1,h_2)\, d(h_1,h_2)$$

$$RO_\gamma(f) := \int_{(A \backslash H_\gamma(\mathbb{A}_F)) \backslash (A_{G,H} \backslash H(\mathbb{A}_F))} f^{-1}(h_1\gamma h_2)\, \chi(h_1 h_2) \frac{d(h_1,h_2)}{dh_\gamma}$$

$A_G \leq Z_G(F \otimes_{\mathbb{Q}} \mathbb{R})$

($H \leq G \times G$, $H \curvearrowright X = G$, ~~then~~ $A_{G,H} := H(\mathbb{A}_F) \cap A_G \times A_G$)

$H = \Delta(G) \Rightarrow A_{G,H} = \Delta(A_G) \Rightarrow \mathrm{rtr}_{H,\chi}(f)$ 가.. $\int_{\Delta(A_G)\Delta(G)(F) \backslash \Delta(G)(\mathbb{A}_F)} K_{\pi(f)}(g,g)\, dg$

$$\simeq \int_{A_G G(F) \backslash G(\mathbb{A}_F)} K_{\pi(f)}(g,g)\, dg = m(\pi)\, \mathrm{Tr}(\pi(f))$$

$\Downarrow$

$H_\gamma = \{ (g,g) \in \Delta(G) : g^{-1}\gamma g = \gamma \}$
$= \Delta(G_\gamma)$. centralizer.

$\Downarrow$

$$(A \backslash H_\gamma(\mathbb{A}_F)) \backslash (A_{G,H} \backslash H(\mathbb{A}_F)) \simeq G_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F) \Rightarrow \tau(\Delta(G_\gamma)) \int_{G_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f(g^{-1}\gamma g) \frac{dg}{dg_\gamma}$$

$\tau(H_\gamma)\, RO_\gamma(f) = \tau(\Delta(G_\gamma)) \int \cdots = \tau(G_\gamma) \cdot O_\gamma(f)$.